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Bilinearization of the non-local Boussinesq equation

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Abstract. A single-field bilinear system generating the so-called non-local Boussinesq equation is constructed. From the bilinearization procedure it can be seen that the associated hierarchy of soliton systems which we construct shares part of the solution set of a hierarchy related to the Kadomtsev–Petviashvili equation. A bilinear representation of the recursion operator for the Kaup hierarchy is essential in the construction and a systematic way of obtaining such a representation from just two-soliton considerations is presented.

1. Introduction

Two well known and powerful tools in soliton theory are the Hirota bilinear operators and the bilinear forms expressed in terms of such operators (Hirota 1980). In the past two decades they have proved to be extremely useful, not only for constructing solutions for a vast set of soliton systems, but also in the investigation of several aspects of integrability which appear in these systems (Hirota 1974, Hietarinta 1987). Another appealing feature of the bilinear operators is that they appear in a natural way in Sato theory (Ohta *et al* 1988), such that bilinear formulations exist for a large variety of integrable equations (Jimbo and Miwa 1983) in $2+1$ dimensions or in $(1+1)$-dimensional reductions thereof. Hence the quite popular belief (or should one call it a conjecture?) that 'all' soliton systems can be 'bilinearized in one way or another'. In the light of this proposition, it is interesting to investigate the 'bilinearizability' of newly discovered soliton systems, especially when, at first sight, such systems appear not to be bilinearizable in any straightforward manner.

Recently, a so-called non-local Boussinesq (NLBq) equation was presented (Lambert *et al* 1994) as a resonance-free alternative to the good Boussinesq equation. The behaviour of the relevant two-soliton solutions, as compared to those of the good Boussinesq equation, is discussed in detail in Lambert *et al* (1994). This NLBq equation is a sech-squared soliton system related through a Bäcklund transformation to Kaup's higher-order water wave equation. Because of the sech-squared nature of the solitons, one expects to find a bilinear form for this equation expressed in a single field, in contrast to the two-field bilinear form which is known for Kaup's equation (bearing in mind KdV–MKdV). In this paper, we will show how one can obtain a bilinear system for this equation, expressed in a single field but using a multidimensional approach. In fact, the system arises naturally when one tries to bilinearize higher-order flows belonging to the hierarchy associated with the NLBq system.

No recursion operator is readily available for this hierarchy, so the first step in our procedure must be the characterization of its higher-order flows. This characterization can be obtained by exploiting the Bäcklund transformation which exists between Kaup's equation and the NLBq system, such as that which maps higher-order Kaup equations to higher-order NLBq equations. Since this Bäcklund-link is best described at the bilinear level, a bilinear description of the Kaup hierarchy is needed.

Starting from the (available) bilinear form of the Kaup equation, it is possible to construct a so-called *canonical bilinear form* (Willox *et al* 1993) for the entire hierarchy associated with this equation, such that it (when expressed in the proper fields) reduces to the action of the recursion operator for the classical Boussinesq (CBq) or Broer–Kaup (BK) systems (which are known to generate the Kaup equation upon elimination of the proper variable). This canonical bilinear form first appears in the work of Liu and collaborators (Liu *et al* 1990) as a bilinearization of the recursion operator for the CBq equation, but no explicit construction was given. In this paper we propose such a construction, using nothing more than the explicit form of the solitary waves and the two-soliton solutions of Kaup's equation. The canonical bilinear form then gives rise to a canonical linear system, the compatibility condition of which will define the higher-order flows of a NLBq hierarchy.

Finally, bilinearization of the first few members of this hierarchy reveals a strong connection with the Kadomtsev–Petviashvili hierarchy: the solutions to the NLBq systems are part of the solution set of a bilinear hierarchy related to the KP hierarchy (for a specific value of the parameter appearing in the definition of the NLBq system, this hierarchy reduces to the KP hierarchy). The bilinear form for the NLBq system itself is found to consist of a pair of bilinear equations, expressed in one dependent but three independent variables.

A straightforward consequence of this bilinear system is the existence of symmetric Wronskian solutions previously found using a trilinear form for Kaup's equation (Matsukidaira *et al* 1990, Satsuma *et al* 1992).

2. The non-local Boussinesq system

The non-local Boussinesq system (Lambert *et al* 1994)

$$\begin{aligned} U_t &= V_x \\ V_t &= \alpha U_x - U_{3x} - (U^2)_x + \left[\frac{V^2 + U_x^2}{U - \alpha/2}\right]_x \end{aligned} \tag{1}$$

(where a subscript nx, mt stands for the derivative $\partial^{(n+m)}/\partial x^n t^m$) was introduced through a direct bilinearization scheme for Kaup's higher-order water wave equation (Kaup 1975)

$$w_{2t} - \alpha w_{2x} + w_{4x} + \tfrac{1}{2}(w_x^2)_t + (w_x w_t + \tfrac{1}{2} w_x^3)_x = 0. \tag{2}$$

In Lambert *et al* (1994) it was shown this equation can be put into bilinear form by setting $w = -2\mathrm{i} \ln F/G$:

$$\begin{aligned} (\mathrm{i}D_t + D_x^2)\ F \cdot G &= 0 \\ (\mathrm{i}D_x D_t - \alpha D_x + D_x^3)\ F \cdot G &= 0. \end{aligned} \tag{3}$$

This system can also be looked upon as the bilinear representation of the classical Boussinesq (CBq) equation (Hirota and Satsuma 1977, Hirota 1985).

The Hirota D-operators used in (3) are defined in the usual manner (Hirota 1980):

$$D_x^p D_t^q\ F \cdot G = \left[(\partial_t - \partial_{t'})^q (\partial_x - \partial_{x'})^p F(x,t) G(x',t')\right]\Big|_{\substack{x' \to x \\ t' \to t}}. \tag{4}$$

Setting $\Psi = F/G$ and $q = 2 \ln G$, the bilinear system (3) linearizes in the following way:

$$\begin{aligned} &\mathrm{i}\Psi_t + \Psi_{2x} + q_{2x}\Psi = 0 \\ &\mathrm{i}\Psi_{xt} + \Psi_{3x} - \alpha\Psi_x + 3q_{2x}\Psi_x + \mathrm{i}q_{xt}\Psi = 0. \end{aligned} \tag{5}$$

The NLBq system (1) is found as the compatibility condition of this linear system expressed in terms of the variables $U = q_{2x}$ and $V = q_{xt}$. It derives its name from the fact that when the V-variable is completely eliminated from the system, the resulting equation in U would be explicitly non-local.

Due to the parity property of the D-operators which follows from the definition (4):

$$D_x^p D_t^q \; F \cdot G = (-)^{(p+q)} D_x^p D_t^q \; G \cdot F$$

and using the invariance of (1) under $t \to -t$ and $V \to -V$, it is easily deduced that the bilinear system (3) for Kaup's equation can be interpreted as a (bilinear) Bäcklund transformation for the NLBq system, i.e. if F and G satisfy (3) then both $q = 2\ln G$ and $\tilde{q} = 2\ln F$ give rise to solutions (U, V) and $(\tilde{U}, \tilde{V})$ of the NLBq system.

Since the linearizing transformation mentioned above depends only on the combinatorial properties of the Hirota D-operators (and is therefore always possible), a simple scheme to produce higher-order flows for the NLBq system becomes available. First one has to find a bilinear expression for the Kaup hierarchy. Linearization of this bilinear expression will then give us an infinite-dimensional linear system, the compatibility condition of which will define the non-local Boussinesq hierarchy.

3. A canonical bilinear form for the Kaup hierarchy

One possible way to construct a bilinear representation for the Kaup hierarchy would be by 'bilinearizing' the (known) recursion operator for Kaup's equation. Such a bilinear system was first proposed by Liu and collaborators (Liu *et al* 1990), unfortunately without any clear indication of the method they used to obtain it.

As a direct bilinearization of this operator turns out not to be a straightforward matter at all, for completeness we shall briefly present an alternative but systematic construction of a bilinear representation of the Kaup hierarchy. The appealing feature of our method is that it only relies on the explicit form of the one- and two-soliton solutions of Kaup's equation.

Let us for simplicity consider the case where $\alpha = 0$. The bilinear form (3) for Kaup's equation then becomes

$$\begin{aligned} &(\mathrm{i}D_{t_2} + D_x^2)\; F \cdot G = 0 \\ &(\mathrm{i}D_x D_{t_2} + D_x^3)\; F \cdot G = 0 \end{aligned} \tag{6}$$

where we denote the Kaup time evolution by t_2.

The first step in our procedure will be to construct the first higher-order flow in the hierarchy (its 'time evolution' will be denoted by the variable t_3). The corresponding bilinear system will be completely characterized by the one- and two-soliton solutions it is required to share with (6):

$$\begin{aligned} &F = 1 \\ &G = 1 + \exp\theta \qquad \theta = kx - \mathrm{i}k^2 t_2 - k^3 t_3 + \tau \end{aligned} \tag{7}$$

$$\begin{aligned} &F = 1 + \exp\theta_1 + K_1 \exp(\theta_1 + \theta_2) \\ &G = 1 + \exp\theta_2 + K_2 \exp(\theta_1 + \theta_2) \end{aligned} \tag{8}$$

where

$$\theta_1 = k_1 x + \mathrm{i}k_1^2 t_2 - k_1^3 t_3 + \tau_1 \qquad K_1 = \frac{k_1^2}{(k_1 + k_2)^2}$$

$$\theta_2 = k_2 x - \mathrm{i}k_2^2 t_2 - k_2^3 t_3 + \tau_2 \qquad K_2 = \frac{k_2^2}{(k_1 + k_2)^2}$$

fixing the t_3 dispersion relation to be $\omega^{(3)} = -k^3$.

Let us now suppose there exists a single bilinear system describing all time evolutions of the hierarchy associated with (2). If we assume the bilinear system (6) to be the t_2 realization of this 'canonical' system, the bilinear equations composing the system for the t_3 case can only consist of a few combinations of D-operators: a first bilinear equation (of weight 3, where the weights of x, t_m are taken to be 1 and m, respectively) can only consist of D_{t_3}, $D_x D_{t_2}$ and D_x^3 terms, whereas its weight-4 partner in the system can only contain $D_x D_{t_3}$, $D_x^2 D_{t_2}$, $D_{t_2}^2$ and D_x^4 terms. Furthermore, if the canonical bilinear system has to be a proper 'bilinearization' of the action of the recursion operator (i.e. of an operator linking two successive flows in the hierarchy), it can easily be seen that contributions of D-operators with an increasing polynomial degree (with increasing order of the flow) are inadmissible. In the present case this restriction rules out the appearance of D_x^3 and D_x^4 terms in the weight-3 and weight-4 bilinear equations, respectively.

Under the above assumptions, we find the following one parameter family of bilinear systems exhibiting the solitary wave solutions (7) of the Kaup equation:

$$\begin{aligned} &(D_x D_{t_2} + \mathrm{i} D_{t_3})\ F \cdot G = 0 \\ &(\beta D_{t_2}^2 + (1-\beta) D_x D_{t_3} - \mathrm{i} D_x^2 D_{t_2})\ F \cdot G = 0. \end{aligned} \tag{9}$$

Now imposing the existence of Kaup-type two-soliton solutions (8), the value of the parameter is restricted to $\beta = -1$, giving rise to the following bilinear system for the evolution of t_3:

$$\begin{aligned} &(\mathrm{i} D_{t_3} + D_x D_{t_2})\ F \cdot G = 0 \\ &(2\mathrm{i} D_x D_{t_3} - \mathrm{i} D_{t_2}^2 + D_x^2 D_{t_2})\ F \cdot G = 0. \end{aligned} \tag{10}$$

A canonical bilinear system valid for all time evolutions is now immediately determined by the same line of reasoning as the one which lead to (10). It has to consist of two bilinear equations: a first (say weight m) bilinear equation only depending on D_{t_m} and $D_x D_{t_{m-1}}$ contributions, complemented by a weight $m+1$ equation consisting of $D_x D_{t_m}$, $D_x^2 D_{t_{m-1}}$ and $D_{t_2} D_{t_{m-1}}$ terms. Moreover, if the systems (6) and (10) have to be the t_2 and t_3 realizations of such a canonical system and if (3) has to be its $\alpha \neq 0$ reduction, then the only possible canonical bilinear system for the Kaup hierarchy is the following ($t_1 = x$):

$$\begin{aligned} &(\mathrm{i} D_{t_m} + D_x D_{t_{m-1}})\ F \cdot G = 0 \\ &(2\mathrm{i} D_x D_{t_m} - \mathrm{i} D_{t_2} D_{t_{m-1}} + D_x^2 D_{t_{m-1}} - \alpha D_{t_{m-1}})\ F \cdot G = 0 \qquad \forall\, m \geqslant 2. \end{aligned} \tag{11}$$

One can easily verify that (11) admits suitable (multiple-time) generalizations of the solutions given in (7) and (8) for the case $\alpha = 0$, whereas in the case $\alpha \neq 0$ it can be checked to exhibit the following one-soliton solution:

$$\begin{aligned} F &= 1 + C \exp\theta \\ G &= 1 + \exp\theta \end{aligned} \tag{12}$$

with

$$\theta = kx + \omega^{(2)} t_2 + \sum_{m=3} \omega^{(m)} t_m + \tau \qquad C = \frac{\mathrm{i}k^2 + \omega^{(2)}}{-\mathrm{i}k^2 + \omega^{(2)}}$$

$$\omega^{(2)} = \pm k\sqrt{\alpha - k^2} \qquad \omega^{(m)} = \omega^{(2)} \omega^{(m-1)}/k$$

and the two-soliton solution

$$\begin{aligned} F &= 1 + C_1 \exp\theta_1 + C_2 \exp\theta_2 + C_1 C_2\ A_{12} \exp(\theta_1 + \theta_2) \\ G &= 1 + \exp\theta_1 + \exp\theta_2 + A_{12} \exp(\theta_1 + \theta_2) \end{aligned} \tag{13}$$

where the θ_j, C_j and $\omega_j^{(m)}$ are defined as above by adding the index j (1 or 2) whenever appropriate, and where the two-soliton coupling constant A_{12} is defined as

$$A_{12} = \frac{k_1^2 + k_2^2 - \frac{2}{\alpha}\left(k_1^2 k_2^2 + \omega_1^{(2)}\omega_2^{(2)}\right)}{(k_1+k_2)^2}.$$

The bilinear system (11) is indeed a canonical bilinear form for the Kaup hierarchy since it generates the hierarchy in the same way as its recursion operator does. Introducing the two dependent variables $w = \partial_x \ln F/G$ and $z = \partial_x^2 \ln FG$, the canonical bilinear system (11) reduces, after elimination of the t_2-dependence using the $m = 2$ case, to the action of the recursion operator for the so-called classical Boussinesq (CBq) hierarchy (Ito 1984):

$$\begin{pmatrix} w \\ z \end{pmatrix}_{t_m} = \mathrm{i}\begin{pmatrix} w + w_x\partial_x^{-1} & 1 \\ \partial_x^2 - \alpha + 2z + z_x\partial_x^{-1} & w \end{pmatrix} \cdot \begin{pmatrix} w \\ z \end{pmatrix}_{t_{m-1}} \qquad \forall\, m \geqslant 2 \tag{14}$$

which is of course known to be a reformulation of the recursion operator for the Kaup hierarchy. Although we only imposed the existence of one- and two-soliton solutions upon the system (11), since its action corresponds to the hereditary recursion operator (14), it is clear that it also allows N-soliton solutions for all values of N. As mentioned before, the system (11) taken at $\alpha = 0$ can be found in a paper by Liu and collaborators (Liu *et al* 1990) but no justification for its actual form was given there, other than the reduction to the recursion operator.

4. The non-local Boussinesq hierarchy

As was sketched at the end of section 2, the next step in our procedure is the linearization of the canonical bilinear system (11). Setting $\Psi = F/G$ and $q = 2\ln G$ in (11), we obtain the following infinite-dimensional linear system:

$$\begin{aligned} &\mathrm{i}\Psi_{t_m} + \Psi_{xt_{m-1}} + q_{xt_{m-1}}\Psi = 0 \\ &2\mathrm{i}\Psi_{xt_m} - \mathrm{i}\Psi_{t_2 t_{m-1}} + \Psi_{2x,t_{m-1}} + (q_{2x} - \alpha)\Psi_{t_{m-1}} + 2q_{xt_{m-1}}\Psi_x \\ &\qquad + \mathrm{i}(2q_{xt_m} - q_{t_2 t_{m-1}})\Psi = 0 \qquad \forall\, m \geqslant 2. \end{aligned} \tag{15}$$

The compatibility condition of this linear system can be expressed in terms of the variables $U = q_{2x}$ and $V = q_{xt_2}$ so as to give

$$\begin{pmatrix} U \\ V \end{pmatrix}_{t_m} = \mathcal{R}(U,V)\cdot\begin{pmatrix} U \\ V \end{pmatrix}_{t_{m-1}} \qquad \forall\, m \geqslant 2 \quad t_1 = x \tag{16}$$

where the operator $\mathcal{R}(U,V)$ is found to be

$$\mathcal{R}(U,V) = \begin{pmatrix} \dfrac{-V}{2U-\alpha} - 2U_x\partial_x^{-1}\dfrac{V}{(2U-\alpha)^2} & 1 + U_x\partial_x^{-1}\dfrac{1}{2U-\alpha} \\ \alpha - \partial_x^2 - 2U + \dfrac{3U_x}{2U-\alpha}\partial_x + \dfrac{U_{2x}}{2U-\alpha} & \dfrac{3V}{2U-\alpha} + V_x\partial_x^{-1}\dfrac{1}{2U-\alpha} \\ -4\dfrac{V^2+U_x^2}{(2U-\alpha)^2} - 2V_x\partial_x^{-1}\dfrac{V}{(2U-\alpha)^2} & \end{pmatrix}. \tag{17}$$

We will refer to the sequence of evolution equations defined in (16) through the action of the operator $\mathcal{R}(U,V)$, as the non-local Boussinesq hierarchy. In a forthcoming paper it will be shown that the operator $\mathcal{R}(U,V)$ is a hereditary recursion operator for the NLBq system and, hence, that the sequence (16) produced by starting from the x-translational symmetry (U_x, V_x) is a hierarchy of commuting flows.

At $m = 2$, the sequence (16) reduces to the NLBq system itself. The first higher-order member of the NLBq hierarchy calculated from (16) and (17) has the following form:

$$\begin{aligned} U_{t_3} &= \alpha U_x - U_{3x} - \tfrac{3}{2}(U^2)_x + \frac{3}{4}\left[\frac{V^2 + U_x^2}{U - \alpha/2}\right]_x \\ V_{t_3} &= \alpha V_x - V_{3x} - 3(UV)_x + \frac{3}{2}\left[\frac{V_x U_x - V U_{2x}}{U - \alpha/2}\right]_x + \frac{3}{4}\left[\frac{V^3 + U_x^2 V}{(U - \alpha/2)^2}\right]_x . \end{aligned} \tag{18}$$

Note that, although the NLBq system (1) can be found as the compatibility condition of the linear system (5), the corresponding bilinear form (3) does not constitute an applicable Bäcklund transformation at the soliton level: due to the absence of a free parameter the N-soliton solution is not mapped into the $(N+1)$-soliton solution in the general case $\alpha \neq 0$. Since this important Bäcklund property is lacking, it might be more convenient to express the NLBq system as a single-field bilinear form.

5. A bilinearization scheme for the NLBq system

The motivation for wanting to bilinearize the NLBq system is threefold. Apart from the obvious challenge such an attempt presents for the Hirota conjecture, and apart from the great advantage such a bilinear form presents when discussing different aspects of integrability (solitons, Lax pairs, etc), the main motivation in this particular case is that one believes there are quite strong indications of the specific nature of such a bilinear form. First of all there is the fact that the NLBq system has sech-squared soliton solutions. This strongly points to the existence of a KdV-type bilinear representation (i.e. expressed in a single field).

Secondly, we know that the bilinear system that generates Kaup's equation acts as a bilinear Bäcklund transformation for the NLBq system. If one compares this bilinear system with other bilinear Bäcklund transformations found in the literature, one immediately notices that it has precisely the form of a Bäcklund transformation derived by means of Hirota's exchange formalism (Hirota 1974, 1980) from a KdV-type bilinear equation. As we will see, a direct bilinearization based on the aforementioned arguments does not work out in any satisfactory way. However, it is possible to circumvent the problems that arise by tackling the bilinearization of more that one member of the NLBq hierarchy at a time.

Let us first consider the NLBq system itself which, as mentioned earlier, has sech-squared soliton solutions

$$U_{sol} = 2\partial_x^2 \ln\left(1 + e^\theta\right) = \tfrac{1}{2}k^2 \operatorname{sech}^2\left(\frac{\theta}{2}\right) \tag{19}$$

where $\theta = -kx + \omega t + \tau$ and $\omega = \pm\, k\, \sqrt{\alpha - k^2}$ with $k^2 < \alpha$.

This suggests that a dependent-variable transformation $U = q_{2x}$, $V = q_{xt_2}$ might give rise to a 'primary equation' expressible as a linear combination of so-called standard polynomials (see, e.g., Lambert *et al* 1994) in the variable $q(x, t_2, t_3, t_4, \ldots)$. Since these standard polynomials can be identified with the action of D-operators on a single-field variable (say G) by setting $q = 2\ln G$, this would lead to a KdV-type bilinear equation generating the NLBq system. However, expressing the NLBq system (1) in terms of the 'primary' variable q (integrating once with respect to x with zero boundary conditions) we get

$$q_{2t_2} = \alpha q_{2x} - q_{4x} - q_{2x}^2 + \frac{q_{xt_2}^2 + q_{3x}^2}{q_{2x} - \alpha/2}. \tag{20}$$

It can easily be seen that equation (20) can never be expressed as a linear combination of standard polynomials and, hence, that it does not immediately bilinearize by setting $q = 2\ln G$.

Nonetheless, written in this explicitly rational form, we notice that the primary NLBq equation (20) is of overall weight 4 (again assigning weights 1 and m to the x and t_m variables, respectively, and weight 2 to the parameter α). Furthermore, inspection of the first equation in the system (18) (defining the t_3 flow in the NLBq hierarchy) tells us that when expressed in terms of the q variable, the resulting total differential with respect to x can be integrated yielding a second weight-4 primary equation:

$$q_{xt_3} = \alpha q_{2x} - q_{4x} - \frac{3}{2}q_{2x}^2 + \frac{3}{4}\,\frac{q_{xt_2}^2 + q_{3x}^2}{q_{2x} - \alpha/2}. \tag{21}$$

Not only is equation (21) of overall weight 4 but it also contains the same rational contribution in q as one finds in (20). This suggests there exists a unique weight-4 equation that is polynomial in q and its derivatives, namely

$$4q_{xt_3} - \alpha q_{2x} + q_{4x} + 3q_{2x}^2 - 3q_{2t_2} = 0. \tag{22}$$

Equation (22) is readily seen to be a combination of standard polynomials and hence to be bilinearizable by setting $q = 2\ln G$, namely

$$\left[4D_xD_{t_3} - \alpha D_x^2 + D_x^4 - 3D_{t_2}^2\right]\ G\cdot G = 0. \tag{23}$$

The crucial remark to make at this point is that in the case $\alpha = 0$ equation (22) is simply a scaled version of the KP equation (Kadomtsev and Petviashvili 1970). The corresponding bilinear form obtained from (23) can be identified with that for the KP equation found in Jimbo and Miwa (1983) after rescaling the independent variables ($t_m \to (-\mathrm{i})^{m-1}\ t_m$).

Leaving equation (20) for what it is and moving up one step in the NLBq hierarchy, it is found by direct computation that both the first equation in the t_4 flow obtained at $m = 4$ from definitions (16) and (17)

$$U_{t_4} = \alpha V_x - V_{3x} - 3(UV)_x + \left[\frac{V_xU_x - VU_{2x}}{U - \alpha/2}\right]_x + \frac{1}{2}\left[\frac{V^3 + U_x^2V}{(U - \alpha/2)^2}\right]_x \tag{24}$$

and the second equation in system (18) (defining the t_3 flow for the V-field) give rise to weight-5 primary equations for q, again exhibiting the same type of rational contributions. Combining these two primary equations one finds a unique weight-5 equation that is polynomial in q and its derivatives, namely

$$q_{xt_4} = \tfrac{2}{3}q_{t_2t_3} + \tfrac{1}{3}\left(\alpha q_{xt_2} - 3q_{2x}q_{xt_2} - q_{3x,t_2}\right). \tag{25}$$

It is again a combination of standard polynomials in q and hence corresponds to the following bilinear form (setting $q = 2\ln G$):

$$\left[3D_xD_{t_4} - 2D_{t_2}D_{t_3} - \alpha D_xD_{t_2} + D_x^3D_{t_2}\right]\ G\cdot G = 0. \tag{26}$$

As in the weight-4 case, equation (25) with α set to zero and $t_m \to (-\mathrm{i})^{m-1}\ t_m$ can be recognized as the weight-5 flow in the KP hierarchy; equation (26) can again be found in Jimbo and Miwa (1983).

These results show clearly that the first two members of the NLBq hierarchy share part of the solution set of nonlinear evolution equations intimately related to the KP equations (or the KP equations themselves in the case $\alpha = 0$) and one is of course tempted to extrapolate this property to the entire NLBq hierarchy, i.e. appropriate linear combinations of weight m equations occuring in the NLBq hierarchy will be identifiable with a generalization (for

$\alpha \neq 0$) of the weight m KP equation. What does this 'conjecture' imply for the corresponding bilinear equations?

When one constructs the bilinear representation of the KP hierarchy, either through Sato theory (Ohta *et al* 1988) or through a Wronskian approach (Nimmo 1989), a certain number of linearly independent bilinear equations are found at each weight level, although only one of them is actually needed as a bilinear form for the KP equation of that weight. The reason for this redundancy (growing with the weight) in bilinear forms is that at each weight level some differential consequences of lower-weight KP equations turn out to be bilinearizable as well. In the case of the NLBq hierarchy, there is an extra flow present compared with the KP case. The t_2 flow is of course defined by the NLBq system itself, whereas in the KP equation t_2 is merely an independent variable. This extra flow opens the possibility of creating a *larger* number of bilinearizable differential consequences of lower-order flows at a given weight level than would be possible in the KP case. We will now show that such a 'supplementary' bilinear equation gives rise to a bilinear form for the NLBq system.

The same procedure as was used to construct bilinearizable combinations of the t_2, t_3 and t_4 flows of the NLBq hierarchy can be used to construct three linearly independent weight-6 bilinear equations, using appropriate combinations of the flows up to t_5. Two of these are again closely related to the KP hierarchy, namely

$$\begin{aligned}\big[144 D_x D_{t_5} + 20 D_x^3 D_{t_3} - 80 D_{t_3}^2 + 45\, D_x^2 D_{t_2}^2 + D_x^6 \\ + 4\alpha^2 D_x^2 - 5\alpha D_x^4 - 68\alpha D_x D_{t_3}\big]\ G\cdot G = 0\end{aligned} \tag{27}$$

$$\begin{aligned}\big[36 D_{t_2} D_{t_4} - 32 D_{t_3}^2 - 4 D_x^3 D_{t_3} + D_x^6 + 9 D_x^2 D_{t_2}^2 \\ + 4\alpha^2 D_x^2 - 5\alpha D_x^4 - 8\alpha D_x D_{t_3}\big]\ G\cdot G = 0\end{aligned} \tag{28}$$

(for $\alpha = 0$ and with $t_m \to (-\mathrm{i})^{m-1}\ t_m$ these can be found in Jimbo and Miwa (1983)).

The third bilinear equation however turns out to be expressible in terms of the variables x, t_2 and t_3 alone:

$$\big[4 D_x^3 D_{t_3} - \alpha D_x^4 + D_x^6 - 3 D_x^2 D_{t_2}^2 + 12\alpha (D_x D_{t_3} - D_{t_2}^2)\big]\ G\cdot G = 0 \tag{29}$$

and does not correspond in any way to a bilinear form that can be found in the KP hierarchy! Since this extra bilinear equation only depends on the t_2 and t_3 flows it should give access to a bilinear representation of the NLBq system; complementing equation (29) with (23) at the same time defines a bilinear representation for the NLBq system and for its first higher-order flow:

$$\begin{aligned}&\big[4 D_x D_{t_3} - \alpha D_x^2 + D_x^4 - 3 D_{t_2}^2\big]\ G\cdot G = 0\\ &\big[4 D_x^3 D_{t_3} - \alpha D_x^4 + D_x^6 - 3 D_x^2 D_{t_2}^2 + 12\alpha (D_x D_{t_3} - D_{t_2}^2)\big]\ G\cdot G = 0.\end{aligned} \tag{30}$$

Indeed, setting $q = 2\ln G$ in (30) we find

$$\begin{aligned}&4 q_{xt_3} - \alpha q_{2x} + q_{4x} + 3 q_{2x}^2 - 3 q_{2t_2} = 0\\ &4(q_{3x,t_3} + 3 q_{2x} q_{xt_3}) - \alpha (q_{4x} + 3 q_{2x}^2) + (q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3)\\ &\qquad -3(q_{2x,2t_2} + q_{2x} q_{2t_2} + 2 q_{xt_2}^2) + 12\alpha (q_{xt_3} - q_{2t_2}) = 0.\end{aligned} \tag{31}$$

Since the first equation in (31) makes q_{xt_3} explicit, the t_3-dependence in the second equation in (31) can be eliminated, yielding the following weight-6 equation:

$$(q_{2x} - \alpha/2)(q_{2t_2} - \alpha q_{2x} + q_{4x} + q_{2x}^2) - q_{xt_2}^2 - q_{3x}^2 = 0 \tag{32}$$

which is simply the numerator of the primary version (20) of the NLBq system. This equation can in turn be used to eliminate the q_{2t_2} term in the first member of the system (31), yielding the primary version (21) of the t_3 flow of the NLBq hierarchy.

A direct consequence of the bilinear system (30) in the $\alpha = 0$ case is the existence of so-called symmetric Wronskian solutions:

$$f = W(\varphi) = \begin{vmatrix} \varphi & \varphi_x & \cdots & \varphi_{(N-1)x} \\ \varphi_x & \varphi_{2x} & \cdots & \varphi_{Nx} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(N-1)x} & \cdots & \cdots & \varphi_{(2N-2)x} \end{vmatrix} \tag{33}$$

with $\varphi_{t_2} = \pm \mathrm{i}\varphi_{2x}$ and $\varphi_{t_3} = -\varphi_{3x}$. When $\alpha = 0$, the first equation in (30) is satisfied since this is the KP equation for which the Wronskian solution (33) is just a particular case of more general Wronskian solutions (Nimmo 1989). The second equation of (30) can be written (using the first equation) as

$$(4D_xD_{t_3} + D_x^4 - 3D_{t_2}^2)f_x \cdot f_x = 0 \tag{34}$$

Because of the symmetric nature of (33) f_x is again a KP-type Wronskian and hence (34) is trivially satisfied.

Solutions of the type (33) were also described in the context of trilinear forms (Matsukidaira *et al* 1990, Satsuma *et al* 1992). It should be noticed however that the trilinear formalism is only applicable in the $\alpha = 0$ case; interesting solition solutions such as $pq = c$ reductions of the KP solitons will be examined (Loris and Willox 1995) using the bilinear system (30), also in the case $\alpha \neq 0$. How one can construct an applicable bilinear Bäcklund transformation for this system will also be shown in Loris and Willox (1995).

6. Concluding remarks

We have thus shown that the NLBq system (1) can be bilinearized in the form (30) by complementing it with the first higher-order flow of its associated hierarchy. To our knowledge, this is the first example of a bilinear form for a soliton system where time variables from higher-order flows have to be used in order to bilinearize a lower member of a hierarchy. In all known examples exactly the opposite procedure is applied: higher-order members of a hierarchy are bilinearized using extra time variables stemming from lower-order members of the hierarchy (Newell 1985).

The bilinear system we have found immediately yields the exact form of the two-soliton solutions reported in Lambert *et al* (1994); an explicit proof, however, of the existence of N-soliton solutions applying a Wronskian technique to this bilinear form, which turns out to be a $pq = c$ reduction of the KP hierarchy, will be reported elsewhere (Loris and Willox 1995).

The bilinear formulation of the first four members of the NLBq hierarchy reveals an intimate link with the KP hierarchy. Previously a similar relationship between the classical Boussinesq and the modified KP-hierarchies was discussed by Sachs (1988) and in the light of our construction of the NLBq hierarchy, these newly discovered ties with the KP hierarchy do not come as a surprise. We believe, however, that a better understanding of the specific reduction procedure from KP to the NLBq system is necessary, especially where the respective solution sets (and in particular the soliton solutions) are concerned. The related works by Hirota (1985, 1986) for the classical Boussinesq hierarchy can be expected to shed some light in these matters. These problems will be the topic of a forthcoming publication.

The proof of the hereditary nature of the recursion operator (17), together with a bi-Hamiltonian formulation of the NLBq system, will also be presented therein.

Finally, we feel it is important to note that the canonical bilinear form for the Kaup hierarchy (or classical Boussinesq hierarchy) we use in our construction of this recursion operator could be obtained (constructively) just by two-soliton considerations.

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